Let $U = u_1\beta + u_0$, $D = d_1\beta + d_0$, with $0 \le u_1, u_0, d_0 < \beta$, $\beta/2 \le d_1 < \beta$.

Let us revisit the proof of Theorem 3 from [1] with the additional assumption $U < D - d_1$, thus $u_1\beta < D - d_1$:

With the same notations as in [1], except that u_2 is u_1 here, u_1 is u_0 here, and $u_0 = 0$:

$$\begin{aligned}
\beta^2\tilde{r} &= u_1\beta K + u_0\beta(\beta^2 - D) + q_0\beta D - D\beta^2 \\
&< (D - d_1)D + (\beta - 1)\beta(\beta^2 - D) + q_0\beta D - D\beta^2 \\
&= (\beta^2 - D)^2 - d_1 D - \beta^3 + D\beta + q_0\beta D \\
&< (\beta^2 - D)^2 + q_0\beta D - d_1 D
\end{aligned}$$

Thus:

$$\tilde{r} < \frac{\beta^2 - D}{\beta^2}(\beta^2 - D) + \frac{D}{\beta^2}q_0\beta - \frac{d_1 D}{\beta^2} \le \max(\beta^2 - D, q_0\beta) - \frac{d_1}{2} \le \max(\beta^2 - D, q_0\beta) - \frac{\beta}{4}.$$

References

[1] Möller, N., and Granlund, T. Improved division by invariant integers. *IEEE Trans. Comput. 60*, 2 (2011), 165–175.